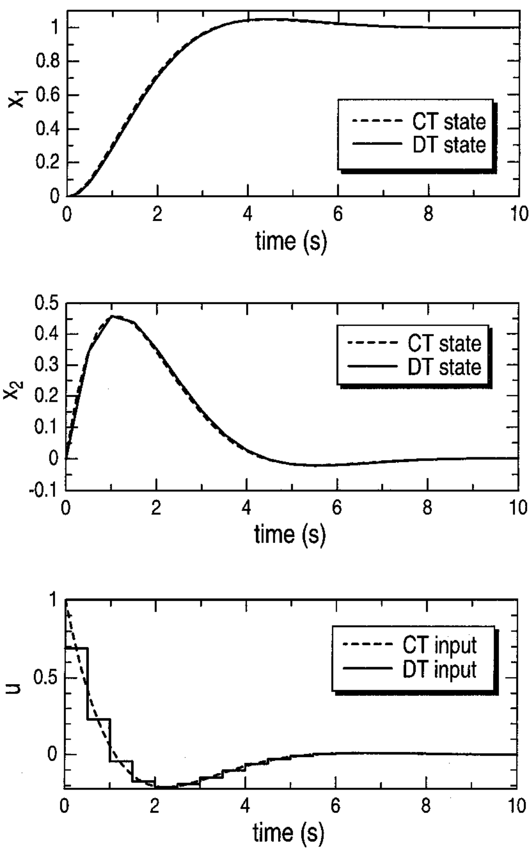

Fig. 1 Time histories of states and inputs.

where $\boldsymbol{E}_o = E(\boldsymbol{A}, T)$ and $\boldsymbol{E}_c = E(\boldsymbol{A}_c, T)$.

Numerical Example

The effectiveness of the redesigned control law is illustrated by a simple numerical example. Let us consider a CT–LTI system described by Eq. (1) with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$

The feedback gain matrix $\boldsymbol{F} = [-1 \quad -1.4]$ gives the closed-loop system a second-order dynamics with a damping ratio 0.7 and a natural frequency 1 rad/s. For a sampling time $T = 0.5$ s and the feedforward gain matrix $\boldsymbol{G} = 1$, the DT gain matrices are computed by Eqs. (10) and (11) as

$$\boldsymbol{F}_d = -[0.6898 \quad 0.1624], \qquad \boldsymbol{G}_d = 0.6898 \tag{15}$$

Figure 1 shows time histories of the states and control input of the CT and DT closed-loop systems for a step input $r(t) = 1$. The DT state responses agree with the CT ones very well, and it can be seen that the DT control input satisfies the PEA in each sampling period.

Conclusions

A digital redesign method for the linear state-feedback control law is derived based on the PEA. The effectiveness of the DT control law is illustrated through a numerical example. The redesigned control law is the same as the one derived by other approaches. As mentioned in the Introduction, however, the PEA also plays an important role in redesign of one-degree-of-freedom controllers and conversion of PAM inputs into PWN inputs. These characteristics of the PEA imply that the PEA is an underlying idea generally applicable to control-input redesign, and the analysis using the mean value theorem explains why it does work.

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Variable Structure Controller Design with Application to Missile Tracking

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Introduction

THIS Note presents the robust autopilot design for missile tracking problem employing variable structure control (VSC) with a power rate reaching law. VSC systems (VSCS) have been extensively studied in the literature over the past 40 years.[1–3] VSC with sliding mode (SM) is a robust nonlinear control for controlling uncertain plants. Robustness is the most important virtue of VSCS. A VSC is characterized by a discontinuous feedback control structure that is switched as the system state crosses certain user specified surfaces called switching surfaces (SS) or sliding surfaces. The control strategy is selected to steer the system state trajectory from any initial state onto the SS and thereafter to constrain the state trajectory motion in the close vicinity of the SS. This results in a reduced-order, surface-dependent sliding motion that is insensitive to disturbance and parameter variations. Assuming all states are available, a linear SS $\sigma[x(t)]$ is used in the present work. Chattering is an undesirable feature of VSC, and it can be eliminated by the continuation method or by the reaching law method.[4]

Missile autopilots are designed to improve stability and performance. The dynamics of the missile are nonlinear, time varying, and coupled.[5] Thukral and Innocenti present VSC design for a missile autopilot that uses both aerodynamic control and reaction jet

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thrusters to achieve high-angle-of-attack maneuvering.[6] The optimal time-varying sliding surface is designed for a missile autopilot by recursively approximating the nonlinear systems as linear time-varying systems.[7] It is well known that the linear time-invariant models developed for design purposes are stable and can deliver satisfactory performance. In the present work a linear model of a missile is considered, and the VSC design of lateral acceleration (LATAX) autopilot for a typical dual input missile with wing and tail as aerodynamic control surfaces is given. The power rate reaching law[4] is used for the reaching phase design to reduce chattering.

Missile Model

The missile considered here is a multi-input, air-to-air missile with cruciform wings. The short-period dynamics of the missile are used for the design of the control system. The linearized equations of motion of the missile in the pitch plane are given by

$$\dot{\alpha} = (Z_\alpha/V)\alpha + q + (Z_{\delta w}/V)\delta_w + (Z_{\delta t}/V)\delta_t \tag{1}$$

$$\dot{q} = M_\alpha \alpha + M_q q + M_{\delta w}\delta_w + M_{\delta t}\delta_t \tag{2}$$

where V is the missile velocity, α is the angle of attack (AOA), q is the pitch rate, and δ_w and δ_t are the deflections of the wing and tail control surfaces, respectively. Z_α, M_α, M_q, and so on, are the aerodynamic stability derivatives. The missile is assumed to have first-order actuator models for wing as well as tail. The output for tracking the LATAX command r is the measured acceleration $a_z = V(\dot{\alpha} - q) + l\dot{q}$, and, hence, $a_z = (Z_\alpha + lM_\alpha)\alpha + lM_q q + (Z_{\delta w} + lM_{\delta w})\delta_w + (Z_{\delta t} + lM_{\delta t})\delta_t$, where l is the distance of the accelerometer ahead of center of gravity. The plant states are $x_p = [\alpha, q, \delta_w, \delta_t]^T$ and input $u = [\delta_{wc}, \delta_{uc}]^T$. The state-space model of the missile in regular form is

$$\dot{x}_p = A_p x_p + B_p u, \qquad y_t = C_p x_p \tag{3}$$

VSC Design for Missile Tracking

In conventional tracking, the controller uses an inner loop with feedback of q to the tail actuator and an outer loop with tracking error in LATAX to the wing actuator. Figure 1 shows the variable structure SM controller for tracking the output y_t for the reference input r. Here the controller is coupled because all states and also the tracking error are fed back to both the actuators, resulting in larger design freedom to achieve better performance.

In the block diagram, the tracking error is $\dot{x}_e = r - y_t$. In the steady state $\dot{x}_e = 0$; thus, the output tracks the reference signal r. The states of the given plant are augmented with x_e, the integral of the tracking error. Hence, the augmented state-space model in regular form, where n_t is the number of outputs to be tracked, is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ B_2\end{bmatrix}u + \begin{bmatrix}b_r\\ 0\end{bmatrix}r \tag{4}$$

where

$$x_1 = \begin{bmatrix}x_e\\ \alpha\\ q\end{bmatrix}, \qquad x_2 = \begin{bmatrix}\delta_w\\ \delta_t\end{bmatrix}, \qquad b_r = \begin{bmatrix}I_{n_t}\\ 0\end{bmatrix}$$

The square matrix B_2 is nonsingular because the input matrix B_p is of full rank. For simplicity, Eq. (4) is rewritten as

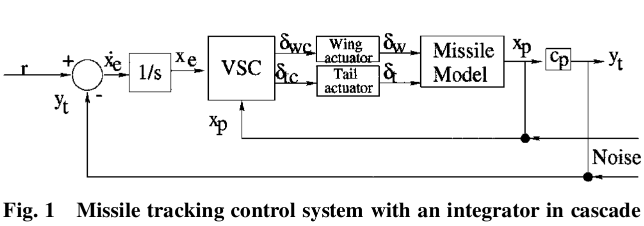

Fig. 1 Missile tracking control system with an integrator in cascade loop.

$$\dot{x} = Ax + Bu + B_r r; \qquad x\in\Re^n, \qquad u\in\Re^m, \qquad r\in\Re^{n_t} \tag{5}$$

SS Design

Here a linear SS of the form $\sigma = Sx(t) = 0$ is used, where S is an $m \times n$ matrix. The existence of the SM implies that $\sigma = 0$ and $\dot{\sigma} = 0$. Hence,

$$\sigma = [S_1 \quad S_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0 \Rightarrow x_2 = -S_2^{-1}S_1 x_1 \tag{6}$$

$$\dot{\sigma} = S\dot{x} = 0 \Rightarrow u_{\text{eq}} = -(SB)^{-1}[SAx + SB_r r] \tag{7}$$

where u_{eq} is the control input necessary to constrain the state trajectory on the SS. Here $S_1\in\Re^{m\times(n-m)}$ and $S_2\in\Re^{m\times m}$ are nonsingular. With this equivalent control and x_2, the dynamics of the reduced-order system moving on the SS can be symbolically written as

$$\dot{x}_1 = [A_{11} + A_{12}F]x_1 + B_r r \tag{8}$$

where

$$F = -S_2^{-1}S_1 \tag{9}$$

The reduced-order dynamics (ROD) has the feedback structure with x_2 as the fictitious control input and A_{12} as the input matrix. Because the regular plant (A, B) is controllable, the pair (A_{11}, A_{12}) is also controllable. Hence, the eigenvalues of the reduced-order system $(A_{11} + A_{12}F)$ can be placed appropriately such that it has the desired characteristics. Here the linear quadratic regulator (LQR) approach is used to find F. Once F is computed, then the matrix S that defines the SS σ can be obtained by Eq. (9), assuming the matrix S_2 to be an identity matrix (without loss of generality). Then, $S_1 = -F$ and, hence, $S = [-F \quad I_{m\times m}]$.

The general form of the VSC control law is $u = u_{\text{eq}} + u_d$, where u_d is the discontinous part of the control law. With the power rate reaching law, $u_d = -(SB)^{-1}k \quad |\sigma|^\beta \text{sgn}(\sigma)$, where k is a symmetric/diagonal positive definite matrix and $0 < \beta < 1$. With the boundary-layer approach, $u_d = \gamma_x + W\sigma$, where $\gamma_x = \rho[\sigma/(\|\sigma\| + d)]$, ρ and d are design parameters greater than zero, and $W > 0$ relates to the rate of convergence. The authors have observed that both the stated control laws give comparable performance for the missile autopilot design. Because of its simplicity, the power rate reaching law is used for the autopilot design in this Note. The control law used for tracking is of the form

$$u = -(SB)^{-1}\left[SAx + SB_r r + k|\sigma|^\beta\,\text{sgn}(\sigma)\right] \tag{10}$$

Reaching Condition Derivation

With the power rate reaching law, a controller is designed that is based on the Lyapunov stability criteria so that the system surely reaches the sliding surface. It can be shown that the control law given in Eq. (10) satisfies this stability criterion and accelerates reaching speed when the state is far from the switching manifold. The power rate reaching law gives a fast and low chattering reaching mode. Consider a Lyapunov function $V(x) = 1/2\sigma^T\sigma$. The existence and reaching condition is satisfied if $\dot{V}(x) < 0$. When Eqs. (5) and (10) are used, $\dot{V}(x)$ reduces to

$$\dot{V}(x) = \sigma^T[-k|\sigma|^\beta\,\text{sgn}(\sigma)] < 0, \qquad \forall\sigma \neq 0$$

Performance Specifications

When the guidance command profile and the hardware limitations are considered, the controller specifications and constraints are chosen as follows:

1) The system response specifications are a) settling time $\leq$0.2 s; b) percentage overshoot $\leq$5%, which implies damping ratio $\xi \approx 0.6$ and bandwidth $\omega_n \approx 30$ rad/s for the dominant mode; and c) AOA $\leq$ 8 deg.

2) The actuator constraints are maximum permissible deflection of wing and tail $\delta_w, \delta_t = \pm 20$ deg.

Simulation Results

Simulation studies are carried out for the linear missile model at different operating conditions, corresponding to different Mach numbers, by using the software packages MATLAB® and SIMULINK. The main objective of the simulation is to validate the autopilot performance and to test the robustness of the methodology. The plant matrices corresponding to the flight condition at sea level and Mach number $M = 1.5$, $\alpha = 2.5$ deg, and $t = 5$ s are

$$A = \begin{bmatrix} -1.3613 & 1 & -0.9383 & -0.2346 \\ -66.665 & -2.2523 & 45.211 & -264.967 \\ 0 & 0 & -64 & 0 \\ 0 & 0 & 0 & -100 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 64 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}^T$$

$$C = [-707.1535 \quad -1.1261 \quad -441.8586 \quad -248.5995]$$

In the computation of F, the weighting matrices Q and R are selected to achieve the desired tracking performance after considerable trial and error as

$$Q = \begin{bmatrix} 0.1 & 0.3 & 0.05 \\ 0.3 & 1 & 0 \\ 0.05 & 0 & 0.75 \end{bmatrix}, \qquad R = \begin{bmatrix} 100 & 0 \\ 0 & 10 \end{bmatrix}$$

The matrix S, which defines the SS, is obtained as

$$S = \begin{bmatrix} 0.0292 & 0.9037 & 0.0375 & 1 & 0 \\ -0.0384 & -3.5487 & -0.3403 & 0 & 1 \end{bmatrix}$$

With this, the closed-loop eigenvalues of the reduced-order system are $\{-70.9227, -13.945 + j\ 6.926$ and $-13.945 - j\ 6.926\}$. The value of $k = \text{diag}([15 \quad 10])$ in Eq. (10) is chosen such that the performance of the system is optimized in terms of reaching time. To reduce chattering, β is selected as 0.95. Figure 2 (solid line) shows the response of the system with process and measurement noise at $M = 1.5$ at a given guidance command input of 20 g. The 3-σ noise levels added to states and a_z in the simulation are given in terms of a set consisting mean, standard deviations $n_{1\sigma}$ (measurement noise), and $n_{2\sigma}$ (state-dependent sensor noise) as follows: 1) q sensor = 0.1 deg/s, 0.5 deg/s, and 0.001 m/s; 2) δ_w and δ_t = 0.1, 0.1, and 0.005 deg; and 3) LATAX output = 0.1, 0.5, and 0.001 m/s^2.

Even after adding noise higher than the maximum expected levels, the system is stable with slight oscillatory response and meets the required specifications. The settling time is found to be 0.15 s,

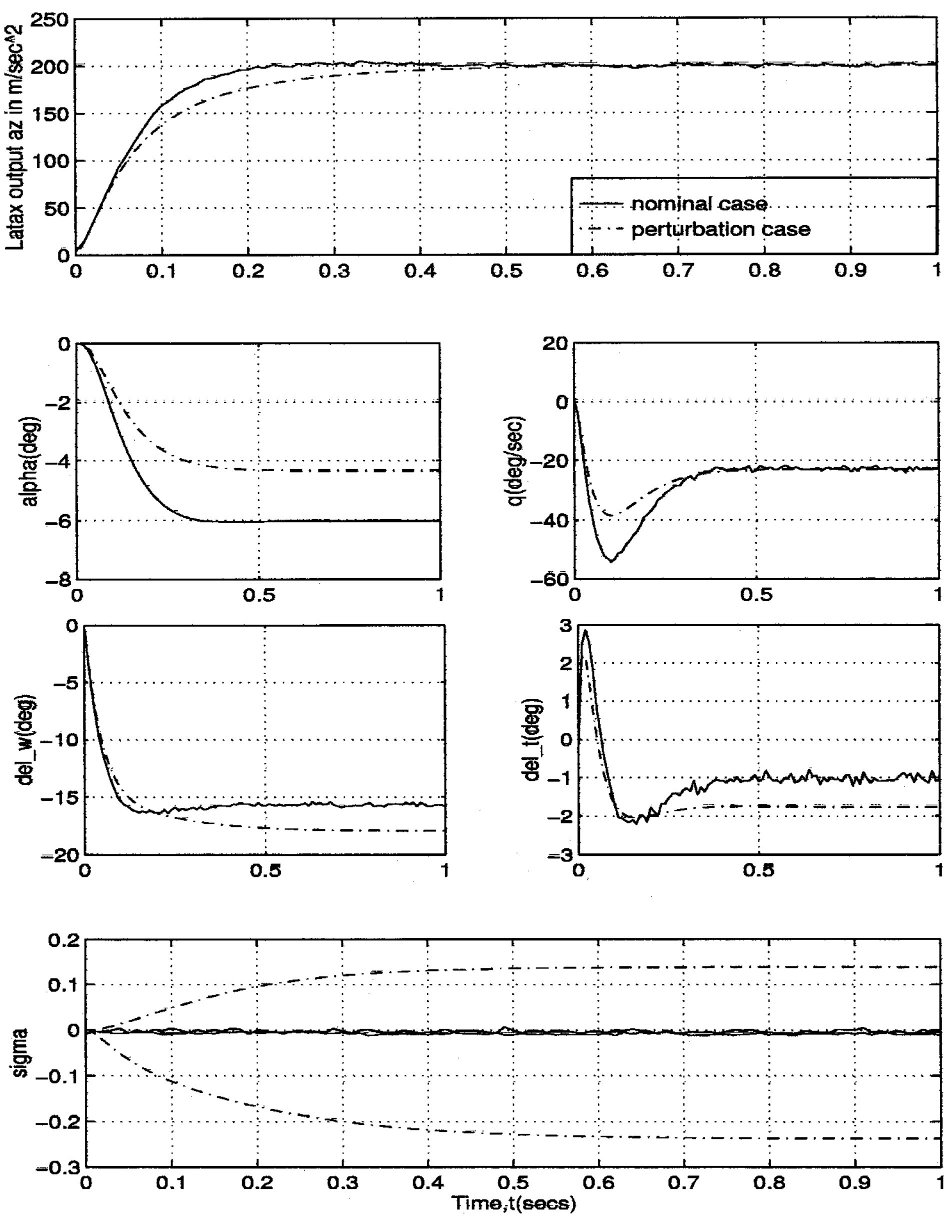

Fig. 2 Simulation results at M = 1.5.

without any overshoot. The closed-loop system is able to minimize the effect of disturbances; only the actuator responses show a small effect of measurement noise. The system responses are well within the maximum limit constraints. As expected, the average value of sigma is nonzero (but small) for the tracking problem. Similar results are obtained for the missile model operating at Mach numbers $M = 1.2$, 2, and 4 with $\alpha = 2.5$ deg and $t = 10$ s, and the performance is found to be satisfactory. An attempt is made to study the robustness of the controller design by testing the model at operating condition $M = 1.2, 1.5$, and 2, whereas the controller is designed for $M = 4$. The same weighting matrix values and k and β work well for the range of Mach numbers considered. The responses corresponding to flight at $M = 1.5$ are given as dotted lines in Fig. 2. The simulation results show that, in the perturbation case, the system is stable and tracks the input without overshoot or steady-state error. The deviation in system responses are well within the limits. The settling time for the specific perturbation case has increased to around 0.4 s, because the dynamic pressure at $M = 1.5$ is substantially smaller than the dynamic pressure at $M = 4$. Because the model at $M = 1.5$ acts like a consistent dynamic model for the controller design at $M = 4.0$, the response demonstrates that the controller is robust.

Conclusions

A pitch control missile autopilot has been designed that uses a dual aerodynamic control input (wing and tail) to track a given LATAX guidance command of 20 g. The autopilot design is based on VSC with power rate reaching law that yields a fast reaching and low chattering. This controller gives satisfactory performance at different Mach numbers (only the simulation corresponding to $M = 1.5$ is shown for brevity), and the tracking error goes to zero in 0.2 s. The perturbation study (though only one case is shown in the paper) endorses robustness of the controller with respect to parameter variations.

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